Lecture notes on MATHEMATICS-II

2nd Semester, B.TECH,ALL BRANCH

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Module-3

1 Vector algebra in 2-space and 3-space

A **vector** is a quantity that is determined by both its magnitude and its direction. A vector is usually given by a initial point and a terminal point. If a given vector v has initial point $P : (x_1, y_1, z_1)$ and terminal point $Q = (x_2, y_2, z_2)$, the three numbers $v_1 = x_2 - x_1, v_2 = y_2 - y_1$, and $v_3 = z_2 - z_1$ are called components of v and we write simply

$$v = (v_1, v_2, v_3).$$

Length of v is

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are added as follows:

$$v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

A scalar c is multiplied to the vector $v = (v_1, v_2, v_3)$ as follows:

$$cv = (cv_1, cv_2, cv_3).$$

Basic properties of vector addition and scalar multiplication

(a) $v + w = w + v$

(b) $(v + w) + z = v + (w + z)$

(c) $v + 0 = 0 + v = v$

(d) $v + (-v) = 0$

(e) $c(v + w) = cv + cw$

(f) $(c + d)v = cv + dv$

(g) $c(dv) = (cd)v$

(h) $1v = v$

(i) $0v = 0$

(j) $(-1)v = -v$

A vector has length 1 is called a unit vector. The unit vectors along the direction X-axis, Y-axis and Z-axis are i, j and k, respectively. In component form $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$. Another popular representation of a vector $v = (v_1, v_2, v_3)$ is $v = v_1 i + v_2 j + v_3 k$.

Inner products of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

General properties of inner products

(a) $(au + bv) \cdot w = au \cdot w + bv \cdot w$ (Linearity)

(b) $u \cdot v = v \cdot w$ (Commutativity)

(c) $u \cdot u \geq 0$. $u \cdot u = 0$ if and only if $u = 0$. (Positivity)

(d) $(u + v) \cdot w = u \cdot w + v \cdot w$ (Distributive)

(e) $|u \cdot v| \leq |u|\,|v|$ (Cauchy-Schwarz inequality)

(f) $|u + v| \leq |u| + |v|$ (Triangle inequality)

Two vectors u and v are orthogonal if $u \cdot v = 0$.

Some special examples of inner products are

$$i \cdot i = 1, j \cdot j = 1, k \cdot k = 1,$$

and

$$i \cdot j = 0, j \cdot k = 0, k \cdot i = 0.$$

Let θ is the angle between u and v, then

$$cos(\theta) = \frac{u \cdot v}{|u|\,|v|}.$$

Applications of inner products

Example 1.1. Work done by a force p on a body giving displacement d is $p \cdot d$.

Example 1.2. Component of a force p in a given direction d is $\frac{p \cdot d}{|p|}$.

Example 1.3 (Orthogonal straight lines in the plane)**.** Find the straight line L_1 through the point $P(1, 3)$ in the xy-plane and perpendicular to the straight line $x - 2y + 2 = 0$.

Let L_1 be the straight line $ax + by = c$, and L_1^* be the straight line $ax + by = 0$. L_1^* passes through the origin and is parallel to L_1. Let us denote the straight line $x - 2y + 2 = 0$ by L_2 and $x - 2y = 0$ by L_2^*. L_2^* passes through the origin and is parallel to L_2. The point $(b, -a)$ lies on L_1^*. Similarly, the point $(2, 1)$ lies on L_2^*. Therefore, L_2^* and L_1^* are perpendicular to each other if $(2, 1) \cdot (b, -a) = 0$, for instance, if$a = 2, b = 1$. Therefore, L_1 is the straight line $2x + y = c$. Since, it passes through $(1, 3)$ $c = 5$. Hence the desired straight line is $2x + y = 5$.

Example 1.4 (Normal vector to a plane)**.** Find a unit vector perpendicular to the plane $4x + 2y + 4z = -7$.

We may write any plane in space as $a \cdot r = a_1x + a_2y + a_3z = c$, where $a = (a_1, a_2, a_3) \neq 0$ and $r = (x, y, z)$. Normalizing, we get $n \cdot r = p$, where $p = \frac{c}{|a|}$ and $n = \frac{a}{|a|}$. This n is the unit vector normal to the given plane. In our example, $n = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Vector product (Cross product)

The vector product or cross product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is the vector $v = (v_1, v_2, v_3) = a \times \times b$, where

$$v_1 = a_2b_3 - a_3b_2, v_2 = a_3b_1 - a_1b_3, v_3 = a_1b_2 - a_2b_1.$$

It is easy to remember the vector product $a \times b$ by the symbolical determinant formula

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Vector product of standard basis vectors

$$i \times j = k, j \times k = i, k \times i = j,$$
$$j \times i = -k, k \times j = -i, i \times k = -j.$$

General properties of vector product

1. $(la) \times b = l(a \times b) = a \times (lb)$
2. $a \times (b + c) = a \times b + a \times c$ (distributive)
3. $(a + b) \times c = a \times c + b \times c$ (distributive)
4. $a \times b = -b \times a$ (anticommutative)
5. $a \times (b \times c) \neq (a \times b) \times c$ (not associative)

Applications of vector product

Example 1.5 (Moment of a force)**.** Let a force p acts on a line through a point A. The moment vector about a pint Q is

$$m = r \times p,$$

where r is the vector whose initial point is Q and the terminal point is A.

Example 1.6. Velocity of a rotating body B rotating with angular velocity w is

$$v = w \times r,$$

where r is the position vector of any point on B referred to a coordinate system with origin 0 on the axis of rotation.

Scalar triple product

The scalar triple product of three vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ is denoted by (a, b, c) and is defined by

$$(a, b, c) = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

An important property of scalar triple product is $a \cdot (b \times c) = (a \times b) \cdot c$.

Theorem 1.7. *Three vectors form a linearly independent set if and only if their scalar triple product is not zero.*

Here are some important facts:

- The modulus of $a \times b$ is the area of a parallelogram with a and b as the adjacent sides.
- The absolute value of the scalar triple product $|(a, b, c)|$ is the volume of the parallelepiped with a, b and c as the concurrent edges.
- The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.
- The area of a triangle is $1/2$ of the area of the parallelepiped.

PROBLEMS

1. Find the area of the parallelogram if the vertices are $(1, 1), (4, -2), (9, 3), (12, 0)$.
2. Find the area of the triangle in space if the vertices are $(1, 3, 2), (3, -4, 2)$ and $(5, 0, -5)$.

2 Vector differential calculus, basic definitions

Definition 2.1 (Gradient)**.** The gradient of a scalar field ϕ is the vector field $\nabla\phi$ given by

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k,$$

whenever these partial derivatives are defined.

Example 2.2. Let $\phi(x, y, z) = x^2 y \cos(yz)$. Then

$$\nabla\phi = 2xy\cos(yz)i + \left(x^2\cos(yz) - x^2yz\sin(yz)\right) j - x^2y^2\sin(yz)k.$$

The gradient field evaluated at a point P is denoted by $\nabla\phi(P)$. For the gradient just computed,

$$\nabla\phi(1, -1, 3) = -2\cos(3)i + [\cos(3) - 3\sin(3)]j + \sin(3)k.$$

Definition 2.3 (Divergence)**.** The divergence of a vector field $F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k$ is the scalar field

$$divF = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

For example, if $F = 2xyi + (xyz^2 - \sin(yz))j + ze^{x+y}k$, then

$$divF = 2y + xz^2 - z\cos(yz) + e^{x+y}.$$

Definition 2.4 (Curl)**.** The curl of a vector field $F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k$ is the vetor filed

$$curlF = \nabla \times F = (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})i + (\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x})j + (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})k.$$

Foe example, if $F = yi + 2xzj + ze^x k$, then

$$curlF = -2xi - ze^z j + (2z - 1)k.$$

Theorem 2.5. *Let ϕ be continuous in its first and second partial derivatives then $curl(grad\phi) = 0$.*

Proof. By direct computation

$$\begin{aligned} \nabla \times \nabla\phi &= \nabla \times (\frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k) \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \\ &= 0. \end{aligned}$$

□

Theorem 2.6. *Let F be a continuous vector field whose components have continuous first and second partial derivatives. Then*

$$div(curl F) = 0.$$

Proof. By direct computation

$$\begin{aligned}
\nabla \cdot (\nabla \times F) &= \frac{\partial}{\partial x}(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) \\
&= 0.
\end{aligned}$$

□

3 Derivatives, directional derivatives

Definition 3.1 (Directional derivative)**.** The directional derivative of a scalar field ϕ at P_0 in the direction of the unit vector $u = ai + bj + ck$ is denoted by $D_u\phi(P_0) = \frac{d}{dt}\phi(x + at, y + bt + z + ct)\big|_{t=0}$.

We usually compute a directional derivative using the following theorem.

Theorem 3.2. *If ϕ is a differentiable function of three variables, and u is a constant unit vector, then*

$$D_u\phi(P_0) = \nabla\phi(P_0) \cdot u.$$

Example 3.3. Let $\phi(x, y, z) = x^2y - xe^z$, $P_0 = (2, -1, \pi)$ and $u = \frac{1}{\sqrt{6}}(i - 2j + k)$. Then the rate of change of $\phi(x, y, z)$ at P_0 in the direction of u is the directional derivative

$$\begin{aligned}
D_u\phi(2, -1, \pi) &= \nabla\phi(2, -1, \pi) \cdot u \\
&= \phi_x(2, -1, \pi)\frac{1}{\sqrt{6}} + \phi_y(2, -1, \pi)\frac{-2}{\sqrt{6}} + \phi_z(2, -1, \pi)\frac{1}{\sqrt{6}} \\
&= \frac{1}{\sqrt{6}}\left([2xy - e^z]_{(2,-1,\pi)} - 2[x^2]_{(2,-1,\pi)} + [-xe^z]_{(2,-1,\pi)}\right) \\
&= \frac{-3}{\sqrt{6}}(4 + e^\pi).
\end{aligned}$$

We will now show that the gradient vector $\nabla\phi(P_0)$ points in the direction of maximum rate of increase at P_0, and $-\nabla\phi(P_0)$ in the direction of minimum rate of increase.

Theorem 3.4. *Let ϕ and its first partial derivatives be continuous in some sphere about P_0, and suppose that $\nabla\phi(P_0) \neq 0$. Then*

1. *At P_0, $\phi(x, y, z)$ has its maximum rate of change in the direction of $\nabla\phi(P_0)$. This maximum rate of change is $\|\nabla\phi(P_0)\|$.*

2. *At P_0, $\phi(x, y, z)$ has its minimum rate of change in the direction of $-\nabla\phi(P_0)$. This minimum rate of change is $-\|\nabla\phi(P_0)\|$.*

Example 3.5. Let $\phi(x, y, z) = 2xz + e^y z^2$. We will find the maximum and minimum rates of change of $\phi(x, y, z)$ from $(2, 1, 1)$. First,

$$\nabla\phi(x, y, z) = 2zi + e^y z^2 j + (2x + 2ze^y)k,$$

so

$$\nabla\phi(P_0) = 2i + ej + (4 + 2e)k.$$

The maximum rate of increase of $\phi(x, y, z)$ at $(2, 1, 1)$ is in the direction of this gradient, and this maximum rate of change is $\sqrt{4 + e^2 + (4 + 2e)^2}$.

The minimum rate of increase of $\phi(x, y, z)$ at $(2, 1, 1)$ is in the direction of $-2i - ej - (4 + 2e)k$, and this minimum rate of change is $-\sqrt{4 + e^2 + (4 + 2e)^2}$.

PROBLEMS

Compute the gradient of the function and evaluate this gradient at the given point. Determine at this point the maximum and minimum rate of change of the function.

1. $\phi(x, y, z) = xyz$; (1,1,1)

2. $\phi(x, y, z) = x^2y - sin(xz)$; $(1, -1, \pi/4)$

3. $\phi(x, y, z) = 2xy + xe^z$; (-2,1,6)

4. $\phi(x, y, z) = cos(xyz)$; $(-1, 1, \pi/2)$

4 Gradient of a scalar field

A real-valued function of three variables is called a scalar field. Depending on the function ϕ and the constant k, the locus of points $\phi(x, y, z) = k$ may form a surface in 3-space. Any such surface is called a **level surface** of ϕ. For example, if

$\phi(x, y, z) = x^2 + y^2 + z^2$ and $k > 0$, then the level surface $\phi(x, y, z) = k$ is a sphere of radius $\sqrt{k}$. If $k = 0$ this locus is just a single point, the origin. If $k < 0$ this locus is empty.

Theorem 4.1. *Let ϕ and its partial derivative be continuous. Then $\nabla\phi(P)$ is normal to the level surface $\phi(x, y, z) = k$ at any point P on this surface at which this gradient vector is nonzero.*

Once we have this normal vector, the **equation of the tangent plane** passing through $P_0 = (x_0, y_0, z_0)$ is obtained as

$$\nabla\phi(P_0) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0.$$

Simplifying, we get

$$\frac{\partial\phi}{\partial x}(P_0)(x - x_0) + \frac{\partial\phi}{\partial y}(P_0)(y - y_0) + \frac{\partial\phi}{\partial z}(P_0)(z - z_0) = 0. \tag{1}$$

The parametric equation of the normal line at P_0 is

$$x - x_0 = \frac{\partial\phi}{\partial x}(P_0)t, y - y_0 = \frac{\partial\phi}{\partial y}(P_0)t, z - z_0 = \frac{\partial\phi}{\partial z}(P_0)t,$$

where t varies over the real line.

Here is an example.

Example 4.2. Consider the (level) surface $\phi(x, y, z) = z - \sqrt{x^2 + y^2} = 0$. This surface is a cone with vertex at $(0, 0, 0)$. The gradient vector is

$$\nabla\phi = -\frac{x}{\sqrt{x^2 + y^2}}i - \frac{y}{\sqrt{x^2 + y^2}}j + k,$$

provided both x and y are nonzero. the point $(1, 1, \sqrt{2})$ is a point on the cone. At this point the gradient vector is $\nabla\phi(1, 1, \sqrt{2}) = -\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j + k$. This is the normal vector to the cone at $(1, 1, \sqrt{2})$. The tangent plane at this point has equation

$$-\frac{1}{\sqrt{2}}(x - 1) - \frac{1}{\sqrt{2}}(y - 1) + z - \sqrt{2} = 0,$$

or

$$x + y - \sqrt{2}z = 0.$$

The normal line through this point has parametric equation

$$x - 1 = -\frac{1}{\sqrt{2}}t, y - 1 = -\frac{1}{\sqrt{2}}t, z - \sqrt{2} = t.$$

Simplifying, we get

$$x = 1 - \frac{1}{\sqrt{2}}t, y = 1 - \frac{1}{\sqrt{2}}t, z = \sqrt{2} + t.$$

PROBLEMS

Find the equations of the tangent plane and normal line to the surface at the given point.

1. $x^2 + y^2 + z^2 = 4$; $(1, 1, \sqrt{2})$

2. $z = x^2 - y^2$; (1,1,0)

3. $x^2 - y^2 + z^2 = 0$; (1,1,0)

4. $3x^4 + 3y^4 + 6z^4 = 12$; (1,1,1)

5 Physical interpretation of divergence and curl of a vector field

Physical interpretation of divergence

Suppose $F(x, y, z, t)$ is the velocity of a fluid at point (x, y, z) and time t. Time plays no role in computing divergence, but it is include here because, normally a velocity vector does depend on time t. The divergence of $F(x, y, z, t)$ at time t is interpreted as a measure of the outward flow or expansion of the fluid from this point.

In case *div* of a vector field is zero, we call it *incompressible*.

Physical interpretation of curl

The angular velocity of a uniformly rotating body is constant times the curl of the tangential linear velocity. In other words

$$\Omega = \frac{1}{2}\nabla \times T,$$

where Ω is the angular velocity and T is the tangential linear velocity.

In case *curl* of a vector field is zero, we call it *irrotational*.

PROBLEMS

1. Let F and G be vector fields. Prove that

$$\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

2. Let ϕ and ψ be scalar fields. Prove that $\nabla \cdot (\nabla\phi \times \nabla\psi) = 0$.

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Module-4

1. Vector Integral Calculus

Vector integral calculus can be seen as a generalization of regular integral calculus. extends integrals as known from regular calculus to integrals over curves, called line integrals , surfaces, called surface integrals , and solids, called triple integrals Vector integral calculus is very important to the engineer and physicist and has many applications in solid mechanics, in fluid flow, in heat problems, and others.

Line Integrals

The concept of a line integral is a simple and natural generalization of a definite integral

$$\int_a^b f(x)dx \tag{1.1}$$

Recall that, in (1.1), we integrate the function $f(x)$, also known as the integrand, from $x = a$ along the x-axis to $x = b$. Now, in a line integral, we shall integrate a given function, also called the integrand, along a curve C in space or in the plane. (Hence curve integral would be a better name but line integral is standard). This requires that we represent the curve C by a parametric representation

$$r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k \ \ (a \leq t \leq b). \tag{1.2}$$

The curve C is called the path of integration. The path of integration goes from A to B. Thus $A : r(a)$ is its initial point and $B : r(b)$ is its terminal point. C is now oriented. The direction from $A$to B, in which t increases is called the positive direction on C. We mark it by an arrow. The points A and B may coincide. Then C is called a closed path. C is called a smooth curve if it has at each point a unique tangent whose direction varies continuously as we move along C. We note that $r(t)$ in (1.2) is differentiable. Its derivative $r'(t) = \frac{dr}{dt}$ is continuous and different from the zero vector at every point of C.

Definition and Evaluation of Line Integrals

A line integral of a vector function $F(r)$over a curve $C : r(t)$ is defined by

$$\int_C F(r)dr = \int_a^b F(r(t)).r'(t)dt \tag{1.3}$$

If we write $dr = [dx, dy, dz]$ then (1.3) becomes

$$\int_C F(r(t)).r'(t)dt = \int_C (F_1dx + F_2dy + F_3dz) = \int_a^b (F_1x' + F_2y' + F_3z')dt. \tag{1.4}$$

Example 1.1. *Find the value of the line integral , whenF$(r) = [-y, -xy]$ and C is the circular arc of unit cir cle in first quardant.*

solution

We may represent C by $r(t) = [cost, sint]$, *where* $0 \leq t \leq \frac{\pi}{2}$. *Then*

$$F(r(t)) = -sinti - costsintj$$

.

$$\int_C F(r(t)).r'(t)dt = \int_0^{\frac{\pi}{2}} (sin^2t - cos^2tsint)dt = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - cos2t)dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - \frac{1}{3}$$

Simple general properties of the line integral

$$\int_C kFdr = kFdr \ (kconstant) \tag{1.5}$$

$$\int_C (F+G)dr = \int_C Fdr + \int_C Gdr \tag{1.6}$$

$$\int_C Fdr = \int_{C1} Fdr + \int_{C2} Fdr \tag{1.7}$$

• Direction-Preserving Parametric Transformations Any representations of C that give the same positive direction on C also yield the same value of the line integral(1.3).

Other Forms of Line Integrals

The line integrals

$\int_C F1dx, \int_C F2dy, \int_C F3dz$

are special cases of(1.3) when $F = F_1iorF_2jorF_3k$, respectively. Furthermore, without taking a dot product as in(1.3) we can obtain a line integral whose value is a vector rather than a scalar, namely,

$$\int_C F(r)dt = \int_a^b F(r(t))dt = \int_a^b [F_1(r(t)), F_2(r(t)), F_3(r(t))]dt. \tag{1.8}$$

Example 1.2. *Integrate*$F(r) = [xy, yz, z]$ *along the helix* $r(t) = [cost, sint, 3t]$ $(0 \leq t \leq 2\pi)$.

solution

$F(r(t)) = [costsint, 3tsint, 3t]$ *integrated with respect to t from* 0 *to* 2π *gives*

$$\int_0^{2\pi} F(r(t)dt = \left[-\frac{1}{2}cos^2t, 3sint - 3tcost, \frac{3}{2}t^2\right]_0^{2\pi} = [0, -6\pi, 6\pi^2].$$

• **Path Dependence**

Path dependence of line integrals is practically and theoretically so important that we formulate it as a theorem. And this section will be devoted to conditions under which path dependence does not occur. The line integral (1.3) generally depends not only on F and on the endpoints A and B of the path, but also on the path itself along which the integral is taken.

Example 1.3. *Show that the differential form under the integral sign of*

$$I = \int_C \left[2xyz^2 + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right]$$

is exact, so that we have independence of path in any domain, and find the value of I from $A:(0,0,1) to B:(1,\frac{\pi}{4},2)$.

Solution

It is exact as

$$(F_3)_y = 2x^2z + \cos yz - yz\sin yz = (F_2)_z$$
$$(F_1)_z = 4xyz = (F_3)_x$$
$$(F_2)_x = 2xz^2 = (F_1)_y$$
$$f = \int F_2 dy = \int (x^2z^2 + z\cos yz)dy = x^2z^2y + \sin yz + g(x,z)$$
$$f_x = 2xz^2y + g_x = F_1 = 2xyz^2, \quad g_x = 0, \quad g = h(z)$$
$$f_z = 2x^2zy + y\cos yz + h' = F_3 = 2x^2zy + y\cos yz. \quad h' = 0$$

Taking $h = 0$ we get $x^2yz^2 + \sin yz$

And value of the integral

$$I = f(1,\frac{\pi}{4},2) - f(0,0,1) = \pi - \sin\frac{\pi}{2} - 0 = \pi + 1$$

Double Integral

Double integrals over a plane region may be transformed into line integrals over the boundary of the region and conversely. This is of practical interest because it may simplify the evaluation of an integral. It also helps in theoretical work when we want to switch from one kind of integral to the other. The transformation can be done by the following theorem.

Example 1.4. *Evalute*

$$\int_0^2 \int_0^4 (x^2 + y^2)dxdy$$

Solution

$$\int_0^2 \int_0^4 (x^2 + y^2)dxdy = \int_0^2 \left[(\frac{x^3}{3} + y^2x)\right]_0^4$$
$$= \int_0^2 (\frac{64}{3} + 4y^2)$$
$$= \left[\frac{64}{3}y + \frac{4y^3}{3}\right]_0^2$$

$$= 64 + 36 = 100$$

Green's Theorem in Plane

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves . Let $F_1(x, y)$ *and* $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy) \qquad (1.9)$$

Here we integrate along the entire boundary C of R in such a sense that R is on the left as we advance in the direction of integration.

Verification of Green's Theorem in the Plane

Let $F_1 = y^2 - 7y$ *and* $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$.

Solution

In (1.9) on the left we get

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int\int_R [(2y+2) - (2y-7)]\, dxdy = 9\int\int_R dxdy = 9\pi$$

Since the circular disk R has area π

We now show that the line integral in (1.9) on the right gives the same value, 9π. We must orient C counterclockwise, say,$r(t) = [cost, sint]$. Then

$$r'(t) = [-sint, cost]$$

, and on$C$,

$$F_1 = y^2 - 7y = sin^2 t - 7sint, \ and \ F_2 = 2xy + 2x = 2costsint + 2cost.$$

.

Hence the result of line integration

$$\oint (F_1 x' + F_2 y')dt = \int_0^{2\pi} [(sin^2 t - 7sint)(-sint) + 2(costsint + cost)(cost)]dt$$

$$= \int_0^{2\pi} (-sin3t + 7sin^2 t + 2cos^2 tsint + 2cos^2 t)dt$$

$$= 0 + 7\pi - 0 + 2\pi = 9\pi$$

Surfaces for Surface Integrals

With line integrals, we integrate over curves in space , with surface integrals we integrate over surfaces in space. Each curve in space is represented by a parametric equation . This suggests that we should also find

parametric representations for the surfaces in space. This is indeed one of the goals of this section. The surfaces considered are cylinders, spheres, cones, and others. The second goal is to learn about surface normals. Both goals prepare us on surface integrals. Note that for simplicity, we shall say "surface" also for a portion of a surface.

Representation of Surfaces

Representations of a surface S in xyz-space are

$$z = f(x, y) or g(x, y, z) = 0. \tag{1.10}$$

For example, $z = +\sqrt{a^2 - x^2 - y^2}$or $x^2 + y^2 + z^2 - a^2 = 0 \ \ (z \geq 0)$ represents a hemisphere of radius $a$and center 0.

Surface Integrals

To define a surface integral, we take a surface S, given by a parametric representation as just discussed,

$$r(u, v) = [(u, v), y(u, v), z(u, v)] = x(u, v)i + y(u, v)j + z(u, v)k \tag{1.11}$$

where (u, v) varies over a region R in the uv-plane. We assume S to be piecewise smooth , so that S has a normal vector

$$N = r_u \times r_v andunitnormalvectorn = \frac{\mathbf{N}}{|N|} \tag{1.12}$$

at every point (except perhaps for some edges or cusps, as for a cube or cone). For a given vector function F we can now define the surface integral over S by

$$\int\int_S FndA = \int\int_R F(r(u, v))N(u, v)dudv \tag{1.13}$$

Here $N = |N|n$ by (1.12), and $|N| = |r_u \times r_v|$ is the area of the parallelogram with sides $r_u$and r_v, by the definition of cross product. Hence

$$ndA = n|N|dudv = Ndudv. \tag{1.14}$$

And we see that $dA = |N|dudv$is the element of area of S. Also $F.n$ is the normal component of F. This integral arises naturally in flow problems, where it gives the flux across S when $F = \rho v$. This means, that the flux across S is the mass of fluid crossing S per unit time. Furthermore, ρ is the density of the fluid and v the velocity vector of the flow, as illustrated by Example 1.5 below. We may thus call the surface integral (1.13) the flux integral.

We can write (1.13) in components, using $F = [F_1, F_2, F_3]$, $N = [N_1, N_2, N_3]$, and $n = [\cos\alpha, \cos\beta, \cos\gamma]$. Here, α, β, γ are the angles between n and the

coordinate axes; indeed, for the angle between n and i, which gives $cos\alpha = ni/|n||i| = ni$, $cos\beta = nj/|n||j|$, and so on. We thus obtain from (1.13)

$$\int\int_S FndA = \int\int_S (F_1 cos\alpha + F_2 cos\beta + F_3 cos\gamma) = \int\int_R (F_1 N_1 + F_2 N_2 + F_3 N_3) dudv \tag{1.15}$$

In (1.15) **we can write cos a** $\cos\alpha dA = dydz$, $\cos\beta dA = dzdx, \cos\gamma dA = dxdy$**. Then** (1.15) **becomes the following integral for the flux:**

$$\int\int_S FndA = \int\int_S (F_1 dydz + F_2 dzdx + F_3 dxdy). \tag{1.16}$$

We can use this formula to evaluate surface integrals by converting them to double integrals over regions in the coordinate planes of the xyz-coordinate system. But we must carefully take into account the orientation of S (the choice of n). We explain this for the integrals of the F_3-terms,

$$\int\int_S F_3 \cos\gamma dA = \int\int_S F_3 dxdy \tag{1.17}$$

If the surface S is given by $z = h(x, y)$ with (x, y) varying in a region R in the xy-plane, and if S is oriented so that $\cos\gamma \geq 0$, then (1.17) **gives**

$$\int\int_S F_3 \cos\gamma dA = \int\int_R F_3(x, y, h(x, y)) dxdy. \tag{1.18}$$

But if $\cos\gamma < 0$**, the integral on the right of** (1.18) **gets a minus sign in front. This follows if we note that the element of area $dxdy$ in the xy-plane is the projection $|\cos\gamma| dA$ of the element of area dA of S; and we have $\cos\gamma = |\cos\gamma|$when $\cos\gamma > 0$, but $\cos\gamma = -|\cos\gamma| when \cos\gamma < 0$. Similarly for the other two terms in** (1.17)**. At the same time, this justifies the notations in** (1.17)**.**

Example 1.5. *Compute the flux of water through the parabolic cylinder $S : y = x^2$, $0 \leq x \leq 2$, $0 \leq z \leq 3$ if the velocity vector is $v = F = [3z^2, 6, 6xz]$, speed being measured in meters per sec. (Generally, $F = \rho v$, but water has the density $\rho = 1g/cm^3 = 1ton/m^3$.)*

Solution

Writing$x = u$**and** $z = v$**, we have** $y = x^2 = u^2$**. Hence a representation of S is**
$S : r[u, u^2, v] \quad (0 \leq u \leq 2, 0 \leq v \leq 3)$.
By differentiation and by the definition of the cross product
$N = r_u \times r_v = [1, 2u, 0] \times [0, 0, 1] = [2u, 1, 0]$.
On S, writing simply $F(S)$ for $F[r(u, v)]$, we have $F(S)[3v^2, 6, 6uv]$. Hence$F(S)N = 6uv^2 - 6$. By rom (3) t integration we thus get

$$\int\int_S F n dA = \int_0^3 \int_0^2 (6uv^2 - 6)dudv$$

$$= \int_0^3 (3u^2v^2 - 6u)|_{u=0}^2 dv = \int_0^3 (12v^2 - 12)dv = (4v^3 - 12)|_{v=0}^3 = 108 - 36 = 72[m^3/sec]$$

Triple Integrals.

A triple integral is an integral of a function $f(x, y, z)$ taken over a closed bounded, three-dimensional region T in space. We subdivide T by planes parallel to the coordinate planes. Then we consider those boxes of the subdivision that lie entirely inside T, and number them from $1\ to\ n$. Here each box consists of a rectangular parallelepiped. In each such box we choose an arbitrary point, say, (x_k, y_k, z_k) in box k. The volume of box $k$we denote by ΔV_k. We now form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y)_k, z_k)\Delta V_k.$$

This we do for larger and larger positive integers n arbitrarily but so that the maximum length of all the edges of those n boxes approaches zero as n approaches infinity. This gives a sequence of real numbers $Jn_1, Jn_2, \cdot$, We assume that $f(x, y, z)$ is continuous in a domain containing T, and T is bounded by finitely many smooth surfaces . Then it can be shown that the sequence converges to a limit that is independent of the choice of subdivisions and corresponding points (x_k, y_k, z_k). This limit is called the triple integral of $f(x, y, z)$ over the region T and is denoted by

$$\int\int\int_T f(x, y, z)dxdydz$$

or by

$$\int\int\int_T f(x, y, z)dV.$$

Triple integrals can be evaluated by three successive integrations. This is similar to the evaluation of double integrals by two successive integrations

Divergence Theorem of Gauss

Triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely. Such a transformation is of practical interest because one of the two kinds of integral is often simpler than the other. It also helps in establishing fundamental equations in fluid flow, heat conduction, etc., as we shall see. The transformation is done by

the divergence theorem, which involves the divergence of a vector function $F = [F_1, F_2, F_3] = F_1 i + F_2 j + F_3 k$**, namely,**

$$divF = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} \tag{1.19}$$

• Divergence Theorem of Gauss (Transformation Between Triple and Surface Integrals)

Let T **be a closed bounded region in space whose boundary is a piecewise smooth orientable surface** S**. Let** $F(x, y, z)$ **be a vector function that is continuous and has continuous first partial derivatives in some domain containing** T**. Then**

$$\int\int\int_T divF dv = \int\int_S F.ndA. \tag{1.20}$$

In components of $F = [F_1, F_2, F_3]$ **and of the outer unit normal vector** $n = [\cos\alpha, \cos\beta, \cos\gamma]$ **of** S **, formula** (1.19) **becomes**

$$\int\int\int_T \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dxdydz = \int\int_S [F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma]\, dA \tag{1.21}$$

$$= \int\int_S [F_1 dydz + F_2 dzdx + F_3 dxdy].$$

Example 1.6. *Evaluate*

$$\int\int_S \left[x^3 dydz + x^2 ydzdx + x^2 zdxdy\right].$$

where S *is the closed surface consisting of the cylinder* $x^2 + y^2 = a^2 (0 \le z \le b)$ *and the circular disks* $z = 0$ *and* $z = b(x^2 + y^2 \le a^2)$.

Solution

$F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z$**. Hence** $divF = 3x^2 + x^2 + x^2 = 5x^2$**. The form of the surface suggest us to introduce polar coordinate**

$$x = r\cos\theta, \quad y = r\sin\theta \quad and \quad dxdydz = rdrd\theta dz$$

we obtain

$$I = \int\int\int_T 5x^2 dxdydz = 5\int_{z=0}^{b}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta rdrd\theta dz$$

$$= 5b\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta rdrd\theta = 5b\frac{a^4}{4}\int_{\theta=0}^{2\pi}\cos^2\theta d\theta = \frac{5}{4}\pi ba^4$$

Further Applications of the Divergence Theorem

Potential Theory. Harmonic Functions

The theory of solution of Laplace equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad (1.22)$$

is called potential theory. A solution of (1.22) **with continuous second-order partial derivatives is called a harmonic function. That continuity is needed for application of the divergence theorem in potential theory, where the theorem plays a key role**

Example 1.7. *The integrands in the divergence theorem are* $divF$ *and* Fn *. If* F *is the gradient of a scalar function, say,* $F = gradf$*, then* $divFdiv(gradf) = \nabla^2 f;$*. Also,* $Fn = nF = ngradf$*. This is the directional derivative of* f *in the outer normal direction of* S*, the boundary surface of the region* T *in the theorem. This derivative is called the (outer) normal derivative of* f *and is denoted by* $\frac{\partial f}{\partial n}$*. Thus the formula in the divergence theorem becomes*

$$\int\int\int_T \nabla^2 f dv = \int\int_S \frac{\partial f}{\partial n} dA$$

- **Let** $f(x, y, z)$ **be a harmonic function in some domain** D **is space. Let** S **be any piecewise smooth closed orientable surface in** D **whose entire region it encloses belongs to** D**. Then the integral of the normal derivative of** f **taken over** S **is zero**

- **Let** $f(x, y, z)$ **be harmonic in some domain** D **and zero at every point of a piecewise smooth closed orientable surface** S **in** D **whose entire region** T **it encloses belongs to** D**. Then** f **is identically zero in** T**.**

- **Let T be a region that satisfies the assumptions of the divergence theorem, and let** $f(x, y, z)$ **be a harmonic function in a domain** D **that contains** T **and its boundary surface** S **. Then** f **is uniquely determined in** T **by its values on** S **.**
- **If the above assumptions are satisfied and the Dirichlet problem for the Laplace equation has a solution in T, then this solution is unique.**

Stokes Theorem (Transformation Between Surface and Line Integrals) **Let** S **be a piecewise smooth oriented surface in space and**

let the boundary of S be a piecewise smooth simple closed curve C. Let $F(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\int\int_S (curl F).ndA = \oint_C F.r'(s)ds \tag{1.23}$$

Here n is a unit normal vector of S and, depending on n, the integration around C . Furthermore, $r' = \frac{dr}{ds}$ is the unit tangent vector and s the arc length of C

In component form formula (1.23) can be written as

$$\int\int_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] dudv \tag{1.24}$$

$$= \oint_{\bar{C}} (F_1 dx + F_2 dy + F_3 dz)$$

where R is the region of the curve $\bar{C}$ in $uv-$plane corresponding to S represented by $r(u, v)$ and $N = [N_1, N_2, N_3] = r_u \times r_v$

Example 1.8. *In this example we verify Stokes theorem*
Given $F = [y, z, x] = yi + zj + xk$ and S is the paraboloid $z = f(x, y) = 1 - (x^2 + y^2)$ $z \geq 0$

Solution

The curve C is the circle $r(s) = [\cos s, \sin s, 0] = \cos si + \sin sj$ it has the unit tangent vector $r'(s) = [-\sin s, \cos s, 0] = -\sin si + \cos sj$. Then the line integral in (1.23) on the right

$$\oint_C F.dr = \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0]\, ds = -\pi$$

Left side of (1.23)

$curl F = [-1, -1, -1]$ *and* $N = grad(z - f(x, y)) = [2x, 2y, 1]$ **so that** $curl F.N = -2x - 2y - 1$

$$\int\int_S curl F.ndA = \int\int_R (2x - 2y - 1)dxdy$$

$$= \int\int_R (-2r\cos\theta - -2r\sin\theta - 1)rdrd\theta$$

$$= \int_0^{1'} \int_0^{2\pi} (-2r\cos\theta - -2r\sin\theta - 1)rdrd\theta$$

$$= 0 + 0 + (\frac{-1}{2})2\pi = -\pi$$

Assignment-I

Evalute the following integral and check for path independence.

1 $F[y^2, x^2], C : y = 4x^2\ from\ \ (0,0)\ to\ \ (1,4)$
2 $F[xy, x^2y^2], C\ \ from\ \ (2,0)\ straight\ to\ \ (0,2)$
3 F as in Prob. 2, C the quarter-circle from (2, 0) to (0, 2) with center (0, 0)
4 $F = [xy, yz, zx], C : r(t) = [2cost, t, 2sint]\ \ from\ \ (2,0,0)\ to\ (2,2\pi,0)$
5 $F = [x^2, y^2, z^2], C : r(t) = [cost, sint, e^t]\ from\ (1,0,1)to(1,0,e^2\pi)$.
6 $F = [x, z, 2y]\ from\ (0,0,0)\ straight\ to(1,1,0),\ then\ \ to\ (1,1,1),\ back\ to\ (0,0,0)$
7

$$\int_0^3 \int_{-y}^{y} (x^2 + y^2) dx dy$$

Evaluate line integral using Green;s theorem

8

$$F = \left[x^2 e^y, y^2 e^x\right]$$

C**: is the rectangle with vertices (0,0)(2,0)(2,3)(0,3)**

9

$$F = [y, -x]$$

C**: is the circle** $x^2 + y^2 = \frac{1}{4}$

10 Find parametric representation of the surface

$$x^2 + y^2 + \frac{1}{4} z^2 = 1$$

Module-5

Fourier Analysis

The aim of the study of Fourier is to express or approximate by sum of simpler trigonometric functions *i.e* the process of decomposing a function into simpler pieces.

Periodic Functions and Fourier Series

A function *f: R→R* is said to be periodic if there exists some positive real number *p* such that *f(x + p) = f(x)* for all real numbers *x*. The smallest real number *p* with this property is the period of the periodic function *f*.

Note: Linear combination of two periodic functions with same period is also a periodic function of that period.

Examples: Some familiar examples of periodic functions are *sine* functions, *cosine* functions, all constant functions etc.

Examples of functions that are not periodic are real exponential functions, non-constant polynomial functions with real coefficients etc.

Trigonometric Series

Trigonometric series are of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{n} X_i\left(a_n \cos nx + b_n \sin mx\right)\text{.......(1)}$$

are needed in the treatment of many physical problems that lead to partial differential equation, for instance, in the theory of sound, heat conduction, electromagnetic waves and mechanical vibrations.

An important advantage of the series (1) is that it can represent very general function with many discontinuities - like the discontinuous "impulse" function of electrical engineering - where as power series derivatives of all orders.

We begin our treatment with some classical calculations that were first performed by Euler. Our point of view is that the function $f(x)$ in (1) is defined on the closed interval $-\pi \le x \le \pi$, and we must find the co-efficients a_n and b_n in the series expansion. It is convenient to assume, temporarily, that the series is uniformly convergent, because this implies that the series can be integrated term by term from $-\pi$ to π.

Since $\int_{-\pi}^{\pi} \cos nx dx = 0$ and $\int_{-\pi}^{\pi} \cos nx dx = 0$ ......(2)

for $n = 1, 2, ...$, the term-by-term integration yields $\int_{-\pi}^{\pi} \cos nx dx = a_0 \pi$

so $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$.......(3)

It is worth nothing here that formula (3) shows that the constant term $\frac{1}{2}a_0$ in (1) is simply the average value of $f(x)$ over this interval. The co-efficient an is found in a similar way. Thus, if we multiply (1) by $\cos nx$ the result is

$$f(x)\cos nx = \frac{1}{2}a_0\cos nx + ...+ a_n\cos^2 nx + ...(4)$$

where the terms not written contain products of the form $\sin mx\cos nx$ or of the from $\cos mx\cos nx$ with $m \neq n$. At this point it is necessary to recall the trigonometric identities

$$\sin mx\cos nx = \frac{1}{2}\left[\sin\left(m+n\right)x + \sin\left(m-n\right)x\right],$$

$$\cos mx\cos nx = \frac{1}{2}\left[\cos\left(m+n\right)x + \cos\left(m-n\right)x\right],$$

$$\sin mx\cos nx = \frac{1}{2}\left[\cos\left(m-n\right)x + \cos\left(m+n\right)x\right].$$

It is now easy to verify that for integral values of m and $n \leq 1$ we have $\int_{-\pi}^{\pi}\sin mx\cos nxdx = 0..........(5)$

and $\int_{-\pi}^{\pi}\cos mx\cos nxdx = 0, m \neq n..........(6).$

These fiats enable us to integrate (4) term by term and obtain $\int_{-\pi}^{\pi}\cos f(x)\cos nxdx = a_n\int_{-\pi}^{\pi}\cos^2 nxdx = a_n\pi,$

so $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos f(x)\cos nxdx.........(7).$

By (3), formula (7) is also valid for $n = 0$; this the reason for writing the constant term in (1) as $\frac{1}{2}a_0$ rather than a_0. We get the corresponding formula for b_n by essentially the same procedure- w multiply (1) through by $\sin nx$, integrate term by term, and use the additional fact that $\int_{-\pi}^{\pi}\sin mx\sin nxdx = 0, m \neq n..........(8).$

This yields $\int_{-\pi}^{\pi}f(x)\sin nxdx = b_n\int_{-\pi}^{\pi}\sin^2 nxdx = b_n\pi,$

so $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx.........(9).$

These calculations show that if the series (1) is uniformly convergent, then the co-efficients a_n and b_n can be obtained from the sum $f(x)$ by means of above formulas. However, this situation is too restricted to be of much practical value, because how do we know whether a convergent trigonometric series?

We don't - and for this reason it is better to set aside the idea of fining the co-efficients a_n and b_n in an expansion (1) that may or not exists, and instead use formulas (7) and (9) to define certain numbers a_n and b_n that are then used to construct the trigonometric series (1). When this is done, these a_n and b_n are called the Fourier co-effiecients of the function $f(x)$, and the series (1) is called the Fourier of $f(x)$.

Just as being a Fourier series does not imply convergence, for a trigonometric series does not imply that it is a Fourier series. For example, it is known that

$$\sum_{n=1}^{\alpha} \frac{\sin mx}{\log(1+n)} ....(10)$$

Converges for ever values of x, and yet this series is known not to be a Fourier series.

This means that the co-efficients in (10) cannot be obtained by applying formulas (7) and (9) to any integral function $f(x)$, not even if we make the obvious choice and take the series.

A function $f(x)$ is said to be periodic if $f(x+p)=f(x)$ for all values of x, where p is a positive constant. Any positive number p with this property is called a period of $f(x)$.

The Fourier series of the function $f(x)=x, -\pi \le x \le \pi$. is

$$2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} ....).........(11)$$

$\sin x$ in (11) has periods $2\pi, 4\pi,.......,$ and $\sin 2x$ has periods $\pi, 2\pi,........$.

It is easy to see that each term of the series (11) has period 2π - in fait, 2π is the smallest period common to all the terms - so the sum also has period 2π.

Convergency:

We being by pointing out that each term of the series

$$f(x) = \frac{1}{2}a_0 + \sum_{1}^{\alpha}(a_n \cos nx + b_n \sin nx).......(1)$$

has period 2π, and therefore, if the function $f(x)$ is to be represented by the sum, $f(x)$ must also have period 2π. Whenever we consider a series like (1), we shall assume that $f(x)$ is initially given on the basic interval $-\pi \le x \le \pi$ or $-\pi \le x \le \pi$, and that for other values of x, $f(x)$ is defined by the periodicity condition

$$f(x+2\pi) = f(x)......(2).$$

In particular, (2) requires that we must always have $f(\pi)=f(-\pi)$. Accordingly, the complete function we consider is the so-called "periodic extension" of the originally given part of the successive intervals of length 2π that lie to the right and left of the basic interval.

Dirichlet's conditions: A function $f(x)$ is said to have satisfied Dirichlet's conditions in the interval (-*L*,*L*), Provided $f(x)$ is periodic, piecewise continuous, and has a finite number of relative maxima and minima in (-*L*,*L*).

Note: The phrase simple distribution (or often jump discontinuity) is used to finite jump at a point to describe the situation where a function has a finite jump at a point $x = x_0$. This means that $f(x)$ approaches finite but different limits from the left side x_0 and from the right side.

If a bounded function $f(x)$ has only a finite number of discontinuities and only a finite. Numbers of maxima and minima, then all the discontinuities are simple. The function defined by

$f(x) = \sin\frac{1}{x}(x \neq 0)$, $f(0) = 0$ has infinitely many maxima near $x = 0$ and the discontinuity at $x = 0$ not simple. The function defined by

$g(x) = x\sin\frac{1}{x}(x \neq 0)\, g(0) = 0$ and $h(x) = x^2 \sin\frac{1}{x}(x \neq 0), h(0) = 0$.

Also have infinitely many maxima near $x = 0$, but both are continuous at $x = 0$ where as only $h(x)$ is differentiable at this point.

The general situation is as follows:

The continuity of a function is not sufficient for the convergence of its Fourier series to the function, and neither it is necessary. That is it is quite possible for a discontinuous function to be represented everywhere by its Fourier series, provided it is relatively well- behaved between the points of discontinuity. In Dirichlet's condition, the discontinuities are simply and the graph consists of a finite number of increasing or decreasing continuous pieces.

Theorem: If a function f is bounded and integrable in $[0,a], a >$ and monotone in $[0,\delta]$, $0 < \delta < a$, then

$$\lim_{n\to\alpha}\int_0^a f\frac{\sin nx}{x}dx = f(0^+)\int_0^\alpha \frac{\sin x}{x}dx.$$

Note: Integrals of the following two forms are called Dirichlet's integrals.

$$\int_0^a f\frac{\sin nx}{\sin x}dx, \int_0^\alpha f\frac{\sin x}{x}dx$$

Theorem: If a function f is bounded periodic with period 2π and integrable on $[-\pi,\pi]$ and piecewise monotonic on $[-\pi,\pi]$, then

$$\frac{1}{2}a_0 + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$$

$$= \begin{cases} \frac{1}{2}\left\{f(x-0) + \frac{1}{2}f(x+0)\right\}, & \text{for } -\pi < x < \pi \\ \frac{1}{2}\{f(\pi-0) + f(-\pi+0)\}, & \text{for } x = \pm\pi, \end{cases}$$

where a_n, b_n are Fourier co-efficents series of *f*, and *x* a point of $[-\pi,\pi]$.

The *mth* partial sum at point ξ,

Proof: Let $\frac{1}{2}a_0+\sum_{n=1}^{m}(a_n\cos nx+b_n\sin nx)$ be the Fourier series of *f*, and x a point of $[-\pi,\pi]$.

The *mth* partial sum at the point ξ,

$$\frac{1}{2}a_0+\sum_{n=1}^{m}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt+\sum_{n=1}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)(\cos nt\cos nx+\sin nt\sin nx)dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\left[1+2\sum_{n=1}^{m}\cos nlt-x\right]dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+t)\left[1+2\sum_{n=1}^{m}\cos nt\right]dt$$

$$\left(\because \text{ For a periodic function of period } 2\pi, \int_{\alpha}^{\beta}fdx=\int_{\alpha+2\pi}^{\beta+2\pi}fdx\right)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0}f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt+\frac{1}{2\pi}\int_{0}^{\pi}f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$=\frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}}f(x-2t')\frac{\sin(2m+1)t'}{\sin t'}dt'+\frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}}f(2t'+x)\frac{\sin(2m+1)t'}{\sin t'}dt'$$

Proceeding to limits when $m\to\alpha$

$$\frac{1}{2}a_0\sum_{n=1}^{\alpha}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{1}{\pi}\left[\frac{1}{2}\pi f(x-0)+\frac{1}{2}\pi f(x+0)\right]$$

$$=\frac{f(x-0)+f(x+0)}{2}\quad x\in[-\pi,\pi]$$

Thus the Fourier series of a (periodic) function f which is bounded, integrable and piecewise monotonic on $[-\pi,\pi]$, converges to $\frac{1}{2}[f(x-0)+f(x+0)]$ at a point *x*, $-\pi<x<\pi$, and (using periodicity of *f*) to $\frac{1}{2}[f(x-0)+f(x+0)]$ at the ends, $\pm\pi$.

Half –range series:

(A) Cosine series: Let $f(x)$ satisfy Dirichlet's condition in $0 \le x \le \pi$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$ is called Fourier cosine series corresponding to $f(x)$ in the interval. The series is equal to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at every x-in $0 < x < \pi$ where $f(x+0)$ and $f(x-0)$ exists and is equal to $f(0+0)$ at $x = 0$ and equal to $f(\pi - 0)$ at $x = \pi$, provided both $f(0+0)$ and $f(\pi - 0)$ exists.

(B) Sine series: If $f(x)$ satisfies Dirichlet's conditions in $0 < x < \pi$, then $\sum_{n=1}^{\alpha} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx$ represents $f(x)$ in Fourier sine series in the interval. The series is equal to $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point x in $0 < x < \pi$ when $f(x+0)$ and $f(x-0)$ exists and when $x = 0$ and $x = \pi$ the sum is zero.

The Fourier series in other intervals:

(A) In $[0, 2\pi]$: If $f(x)$ satisfies Dirichlet's conditions in $0 \le x \le 2\pi$, then the sum of the series $\frac{1}{2}a_0 + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin mx)$, where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx \\ \text{and } b_n &= \frac{1}{\pi}\int_0^{d\pi} f(x)\sin mxdx \end{aligned}\right\} n \le 1.$$

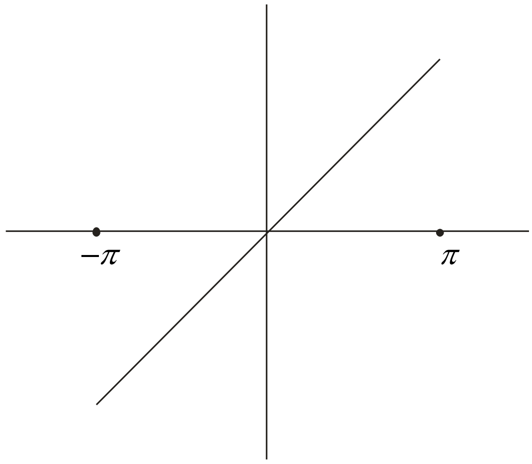
is $\dfrac{f(x_0+0) + f(x_0-0)}{2}$ at any point x_0 in $0 \le x \le 2\pi$ and is $\dfrac{f(x_0+0) + f(x_0-0)}{2}$ at $x = 0$ 2π and is periodic with period 2π.

(Follow from Mallik & Arrora)

Ex.1: If the trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$ is uniformly convergent on $[-\pi, \pi]$ and if $f(x)$ be its sum then prove that it is the Fourier series of $f(x)$ in $[-\pi, \pi]$.

Ex.2: Expand $f(x)=x$ in Fourier series in the interval $-\pi \le x \le \pi$. What soes the series represent for other values of x? what is sum of the series for $x=\pm\pi$ and $x=0$?

Sol: Obviously the function $f(x)=x$ is bounded and integrable on $-\pi \le x \le \pi$, since it is continuous there. Further $f^{/}(x)=1>0$ indicates that $f(x)$ is monotonic increasing the entire interval. Thus $f(x)$ satisfies Dirichlet's conditions on $[-\pi,\pi]$.



Hence the Fourier series Corresponding to $f(x)=x$

is $\dfrac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$

where $a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = 0, a_0=\dfrac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0.$

Since $x\cos nx$ and x are odd functions, and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin mx dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin mx dx$$

Since $x\sin mx$ is even. Thus

$$b_n=\frac{2}{\pi}\left[-x\frac{\cos x}{n}\right]_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}\frac{\cos nx}{n}dx$$

$$=-\frac{2}{n}\cos nm=\begin{cases}-\dfrac{2}{n}, & n \text{ is even}\\ \dfrac{2}{n}, & n \text{ is odd}\end{cases}$$

Hence $f(x)=x$ generates Fourier series in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n\cos nx+b_n\sin nx)=\sum_{n=1}^{\alpha} b_n \sin nx$$

$$=b_1\sin nx + b_2 \sin 2x + b_3 \sin 3x + ...$$

$$=2\left\{\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}.........\right\}.....(1)$$

2nd part: For other values of x it converges to the periodic extension.

3rd part: At $x\pm\pi$, the sum of the series (1) is

$$\frac{f(-\pi+0)+f(-\pi+0)}{2}=\frac{-\pi+\pi}{2}=0.$$

At $x=0$, the sum of the series (1) is $f(0)=0$, (since $f(x)$ is continuous at $x=0$).

Ex.3: Expand Fourier series $x+x^2$ on $-\pi \le x \le \pi$ and deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^3}+.......$$

Sol: Let $f(x)=x+x^2$ on $-\pi \le x \le \pi$. We may define $f(x)$ at $x=\pm\pi$ arbitrary. Let us take $f(x)=x+x^2$ on $-\pi \le x \le \pi$ and $f(\pi)=f(-\pi)$.

Now f is bounded and integrable on $[-\pi,\pi]$. Further $f'(x)<1+2x$, so that $f'(x)<0$ for $x<-\frac{1}{2}$ and $f'(x)<0$ for $x<-\frac{1}{2}$. Thus $f(x)$ is monotonic decreasing on $-\pi\le x\le-\frac{1}{2}$ and monotonic increasing on $-\frac{1}{2}\le x\le\pi$ whereby f is piecewise monotonic on $-\pi\le x\le\pi$. Hence $f(x)$ satisfies Dirichlet's conditions on $[-\pi,\pi]$. Now

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx dx=\frac{4}{n^2}\cos n\pi=\begin{cases}\frac{4}{n^2},\ n\text{ even}\\ -\frac{4}{n^2},\ n\text{ odd.}\end{cases}$$

Similarly $b_n=-\frac{2}{n}$ where n is even and $b_n=\frac{2}{n}$ when n is odd. Thus

$$x+x^2 \square \frac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{\pi^2}{3}-4\left\{\frac{\cos nx}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}.........\right\}+2\left\{\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}.........\right\}$$

And $x+x^2$ is a continuous function; hence on $-\pi\le x\le\pi$,

$$x+x^2=\frac{\pi^2}{3}-4\left\{\frac{\cos nx}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}.........\right\}+2\left\{\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}.........\right\}$$

At $x=\pm\pi$, the sum of the series

$$=\frac{1}{2}\{f(-\pi+0)+f(-\pi+0)\}=\frac{1}{2}(-\pi+\pi^2+\pi+\pi^2)=\pi^2$$

$$\therefore \pi^2=\frac{\pi^2}{3}-4\left(-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}.......\right)$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}....=\frac{\pi^2}{6}.$$

Ex.4: Expand in a series of sines and cosines of multiple of x, the function.

$$f(x)=\begin{cases}x-\pi,-\pi<x<0\\ x-\pi,0<x<\pi.\end{cases}$$

Sol: See that $f(x)$ is not defined at $x=0$, $\pi,-\pi$ where it can be defined in any manner. Let us take $f(x)=-\pi$ at $x=0$ and $f(x)$=0 at $x=\pm\pi$. Obviously the function $f(x)$ is bounded in $-\pi\le x\le\pi$. Obviously the function $f(x)$ is bounded in $-\pi\le x\le\pi$, -2π and π being the bounds. Moreover, $f(x)$ is monotone increasing on $-\pi\le x\le 0$ and $f(x)$ is monotone decreasing on $0<x<\pi$ whereby

$f(x)$ is piecewise monotone on $-\pi \le x \le \pi$. Hence $f(x)$ satisfies Dirichlet's conditions on $[-\pi, \pi]$.

$$\text{Now, } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-\pi)dx + \int_{0}^{\pi}(\pi - x)dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2} - \pi x\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\left(\frac{x^2}{2} - \pi^2\right) + \left(\pi^2 - \frac{x^2}{2}\right)\right\} = -\pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos xdx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(x-\pi)\cos nxdx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nxdx\right\}$$

$$= \frac{1}{\pi}\left\{\left[(x-\pi)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \left[(x-\pi)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{-\pi}^{0}\right\}$$

$$= \frac{2}{n^2\pi}(1 - \cos n\pi) = \frac{2\{1-(-1)^n\}}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(x-\pi)\sin mxdx + \int_{-\pi}^{0}(x-\pi)\sin nxdx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-(x-\pi)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \left[-(x-\pi)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}\right\}$$

$$= \frac{2(1-\cos n\pi)}{n} = \frac{2\{1-(-1)^n\}}{n}$$

The Fourier series corresponding to $f(x)$ on $-\pi \le x \le \pi$ is then

$$-\frac{\pi}{2} + \sum_{n=1}^{\alpha}\frac{2\{1-(-1)^n\}}{n^2}\cos nx + \sum_{n=1}^{\alpha}\frac{2\{1-(-1)^n\}}{n}\sin nx.$$

$$= -\frac{\pi}{2} + 4\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots.\right\} + 4\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\ldots.\right\}\ldots..(1)$$

At $x = \pm\pi$, the sum of the series (1) is $\dfrac{f(-\pi+0)+f(\pi-0)}{2} = -\pi$. At $x = 0$, the sum of the series (1) is $\dfrac{f(0+0)+f(0-0)}{2} = \dfrac{\pi-\pi}{2} = 0$.

Ex.5: Represent $f(x)$ where $f(x)=\cos kx$ on $-\pi \le x \le \pi$ (K not being an integer) in Fourier series.
Deduce that

(i) $\pi \cot k\pi = \frac{1}{k} + 2k\sum_{n=1}^{\alpha} \frac{1}{k^2-n^2}$

(ii) $\frac{\pi}{\sin k\pi} = \sum_{n=0}^{\alpha}(-1)^n \left\{\frac{1}{n+K} + \frac{1}{n+1-K}\right\}.$

Sol: Obviously the function $f(x)=\cos Kx$ is bounded and integrable and piecewise monotonic in the interval $-\pi \le x \le \pi$. Therefore monotone in the interval $-\pi \le x \le \pi$. Therefore $f(x)$ satisfies Dirichlet's conditions on $[-\pi, \pi]$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos Kxdx = \frac{2\sin K\pi}{K\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos Kx \cos nxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos Kx \cos nxdx \left[\text{as } \cos Kx \cos nx \text{ ia an even function}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \{\cos(k+n)x + \cos(K-n)x\}dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(K+n)x}{K+n} + \frac{\sin(K-n)x}{K-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(K+n)x}{K+n} + \frac{\sin(K-n)x}{K-n}\right]$$

$$= \frac{1}{\pi}(-1)^n \sin K\pi . \left(\frac{2k}{k^2-n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos Kx \sin xdx = 0$$

$\left[\text{as } \cos Kx \sin nx \text{ ia an odd function}\right]$

The Fourier series corresponding to $f(x)$ on $-\pi \le x \le \pi$ is then

$$\frac{\sin K\pi}{K_n} + \sum_{n=1}^{\alpha} \frac{2K(-1)^n}{\pi(k^2-n^2)} \sin nK\pi \cos nx ......(1)$$

(i) At $x = \pm\pi,$ the sum of the series (1) is $\frac{f(-\pi+0)+f(\pi-0)}{2}$

$= \cos K\pi.$

$$\therefore \cos K\pi = \frac{\sin K\pi}{K\pi} + \sum_{n=1}^{\alpha} \frac{2K(-1)}{\pi(K^2-n^2)} \sin K\pi \cos n\pi$$

or, $\pi \cot K\pi = \frac{1}{K} + 2K\sum_{n=1}^{\alpha} \frac{1}{K^2-n^2}$

(ii) At $x=0$, the sum of the series is 1,

$$\therefore 1 = \frac{\sin K\pi}{K\pi} + \sum_{n=1}^{\alpha} \frac{2K(-1)^n}{\pi(K^2 - n^2)} \sin K\pi$$

$$\frac{\sin K\pi}{K\pi} = \frac{1}{K} + \sum_{n=1}^{\alpha} \frac{2K(-1)^n}{(K^2 - n^2)} = \frac{1}{K} + \sum_{n=1}^{\alpha} (-1)^n \left\{ \frac{1}{K-n} + \frac{1}{K-n} \right\}$$

$$= \frac{1}{K} + \sum_{n=1}^{\alpha} (-1)^n \left\{ -\frac{1}{K-n} + \frac{1}{K-n} \right\}$$

$$= \frac{1}{K} - \left(-\frac{1}{1-K} + \frac{1}{1+K} \right) + \left(-\frac{1}{2-K} + \frac{1}{2+K} \right) - \left(-\frac{1}{3-K} + \frac{1}{3+K} \right) + \ldots\ldots\ldots$$

$$= \left(\frac{1}{K} + \frac{1}{1-K} \right) - \left(\frac{1}{1+K} + \frac{1}{2-K} \right) + \ldots\ldots\ldots\ldots$$

$$= \sum_{n=0}^{\alpha} (-1)^n \left\{ \frac{1}{1+K} + \frac{1}{n+1+K} \right\}$$

Note: A half range Fourier series is a Fourier series defined on an interval [*0,L*] instead of the more common [-*L,L*]. The function f(x), x ϵ [*0,L*] can be extended to [-*L,0*] for an even or odd function *f(x)*. Hence, the even extension generates the *cosine* half range expansion and the odd extension generates the *sine* half range expansion.

ASSIGNMENT

Q 1: Let

$$f(x) = x, 0 \le x \le \frac{\pi}{2},$$

$$= \pi - x, \frac{\pi}{2} \le x \le \pi,$$

$$= -f(-x), -\pi \le x \le 0.$$

verify that f satisfies Dirichlet's condition on $[-\pi, \pi]$. Obtain the Fourier series for f in $[-\pi, \pi]$.

Q 2: Show that Fourier series corresponding to x^2 on $-\pi \le x \le \pi$ is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\alpha} (-1) \frac{\cos nx}{n^2}$ and hence deduce that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}, 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\ldots = \frac{\pi^2}{12}; 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$$

Q 3: If

$$f(x) = -x \ \text{ for } -\pi \le x \le \pi$$
$$= 0 \quad \text{ for } \ 0 \le x \le \pi$$

Then show that Fourier series corresponding to $f(x)$ on $-\pi \le x \le \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\alpha}\frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\alpha}(-1)^2\frac{\sin nx}{n}$$

Q 4: prove that the even function $f(x) = |x|$ on $-\pi \le x \le \pi$ has a cosine in Fourier's form as

$|x| \square \dfrac{\pi}{2} - \dfrac{4}{\pi}\left\{\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \dots\dots\right\}$ Apply Dirichlet's conditions of convergences to show that he series converges to $|x|$ throughout $-\pi \le x \le \pi$. Also show that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \dots\dots = \dfrac{\pi^2}{8}$.

Q 5: Show that e^{ax} on $-\pi \le x \le \pi$ represents

$$e^{ax}\frac{e^{a\pi} - e^{-a\pi}}{\pi}\left\{\frac{1}{2a} + \sum_{n=1}^{\alpha}\frac{(-1)}{n^2 + a^2}(a\cos nx - n\sin nx)\right\}.$$

Q 6: Show that on $-\pi \le x \le \pi$, $\dfrac{\pi}{2\sinh \pi}e^x = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\alpha}(-1)^n\frac{(\cos nx - n\sin nx)}{1+n^2}$

Q 7: Find a Fourier series representing $f(x)$ on $-\pi \le x \le \pi$ when

$$f(x)0, \quad -\pi \le x \le \pi$$
$$= \frac{1}{4}\pi x, \quad 0 < x < \pi.$$

and deduce that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \dots = \dfrac{\pi^2}{8}$.

Q 8: Find the Fourier series of the periodic function f with period 2π defined as follows:

$$f(x) = 0, \quad \text{for } -\pi \le x \le \pi$$
$$= x, \quad \text{for } \ 0 < x < \pi.$$

what is the sum of the series at $x = 5\pi$? Hence deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots\dots = \dfrac{\pi^2}{8}$.

Q 9: What do you mean by the Fourier service of a function f which is defined, bounded, and integrable on $[-\pi, \pi]$. Find the Fourier cosine which represents the periodic function $f(x) = x$ in $0 < x < \pi$.

Q 10: Expand the function $|x|$ in as Fourier series on $[-1,1]$.

Q 11: Expand $f(x) = \begin{cases} \sinh \pi x, & 0 < x < \dfrac{1}{2} \\ 0, & \dfrac{1}{2} < x < 1. \end{cases}$

In a series of the form $\sum_{n=1}^{\alpha} a_n \sin n\pi x$ and deduce that $\sum_{n=1}^{\alpha} \frac{1}{n^2+(n-1)^2} = \frac{\pi}{2}\tanh\frac{\pi}{2}$.

Q 12: Expand $f(x)$ in Fourier sine series on $0 \le x \le \pi$.
where

$$f(x) = \frac{\pi x}{4}, \quad 0 \le x < \frac{\pi}{2}$$

$$= \frac{\pi}{2}(\pi - x), \quad \frac{\pi}{2} < x < \pi.$$

Q 13: Find the Fourier expansion for $f(x)$ which is periodic with period 2π and which on $0 \le x \le 2\pi$ is given by $f(x) = x^2$.
Find the sum of the series at $x = 4\pi$ and hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$.

Q 14: If a_n and b_n are the Fourier co-eeficients of the function $f(x)$ defined

$$\frac{a_0}{2} + \sum_{K=1}^{n}(a_K \cos Kx + b_K \sin Kx)$$

by the interval $-\pi \le x \le \pi$ show that

$$= \int_{-\pi}^{\pi} \frac{f(x+t)}{2\pi} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} dt.$$